

# TOWARD A CLARITY OF THE EXTREME VALUE THEOREM

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ABSTRACT. We apply a framework developed by C. S. Peirce to analyze the concept of *clarity*, so as to examine a pair of rival mathematical approaches to a typical result in analysis. Namely, we compare an intuitionist and an infinitesimal approaches to the extreme value theorem. We argue that a given pre-mathematical phenomenon may have several aspects that are not necessarily captured by a single formalisation, pointing to a complementarity rather than a rivalry of the approaches.

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## 1. INTRODUCTION

German physicist G. Lichtenberg (1742 - 1799) was born less than a decade after the publication of the cleric George Berkeley's tract *The Analyst*, and would have certainly been influenced by it or at least aware of it. Lichtenberg wrote:

The great artifice of regarding small deviations from the truth as being the truth itself [which the differential calculus is built upon]<sup>1</sup> is at the same time the foundation of wit, where the whole thing would often collapse if we were to regard these deviations in the spirit of philosophic rigour (Lichtenberg 1765-1770 [62, Notebook A, 1, p. 21]).

The “truth” Lichtenberg is referring to is that one cannot have

$$(dx \neq 0) \wedge (dx = 0),$$

namely infinitesimal errors are strictly speaking not allowed and, philosophically speaking, equality must be true equality and only true equality. The allegation Lichtenberg is reporting as fact is Berkeley's contention that infinitesimal calculus is based on a logical inconsistency. The “deviation from the truth” refers to the so-called infinitesimal error that allegedly makes calculus possible.

**1.1. A historical re-appraisal.** Recently the underlying assumption of Lichtenberg's claim, namely the inconsistency of the historical infinitesimal calculus, has been challenged. On Leibniz, see Knobloch (2002 [52]), (2011 [53]); Katz & Sherry (2013 [46]); Guillaume (2014 [35]). The present text continues the re-appraisal of the history and philosophy of mathematical analysis undertaken in a number of recent texts. On Galileo, see Bascelli (2014 [3]). On Fermat, see Katz, Schaps & Shnider (2013 [44]) and Knobloch (2014 [54]). On Euler, see Reeder (2012 [70]) and Bair et al. (2013 [1]), (2014 [2]). On Cauchy, see Borovik & Katz (2012 [16]) and others. Part of such re-appraisal concerns the place of Robinson's infinitesimals in the history of analysis.

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<sup>1</sup>We have reinstated (in brackets) Lichtenberg's comment concerning the differential calculus, which was inexplicably deleted by Penguin's translator R. J. Hollingdale. The German original reads as follows: “Der große Kunstgriff kleine Abweichungen von der Wahrheit für die Wahrheit selbst zu halten, worauf die ganze Differential Rechnung gebaut ist, ist auch zugleich der Grund unsrer witzigen Gedanken, wo oft das Ganze hinfallen würde, wenn wir die Abweichungen in einer philosophischen Strenge nehmen würden.”

We will examine two approaches to the extreme value theorem, the intuitionistic (i.e., relying on intuitionistic logic) and the hyperreal, from the point of view of Peirce's *three grades of clarity* (Peirce [69]).

It is sometimes thought that Robinson's treatment of infinitesimals (see [71]) entails excursions into advanced mathematical logic that may appear as baroque complications of familiar mathematical ideas. To a large extent, such baroque complications are unnecessary. The 1948 ultrapower construction of the hyper-real fields by E. Hewitt [40], combined with J. Łoś's theorem [65] from 1955 (whose consequence is the transfer principle), provide a framework where a large slice of analysis can be treated, in the context of an infinitesimal-enriched continuum. Such a viewpoint was elaborated by Hatcher [37] in an algebraic context and by Lindstrøm [64] in an analytic context (see Section 10). We seek to challenge the received wisdom that a modern infinitesimal approach is necessarily baroque.

Laugwitz authored perceptive historical analyses of the work of Euler and Cauchy (see e.g., [60]), but was somewhat saddled with the numerous variants of the  $\Omega$ -calculus that he developed jointly with Schmieden (1958 [77]). As a result, he did not fully stress the simplicity of the ultrapower approach. Synthetic Differential Geometry (following Lawvere), featuring nilsquare infinitesimals, is based on a category-theoretic framework; see Kock [22] and J. Bell [6], [7].

**1.2. Practice and ontology.** To steer clear of presentism in interpreting classical infinitesimalists like Leibniz and Euler, a crucial distinction to keep in mind is that between syntax/procedures, on the one hand, and semantics/ontology, on the other; see Benacerraf (1965 [8]) for the dichotomy of practice *vs* ontology. Euler's inferential moves involving infinitesimals and infinite numbers find good proxies in the *procedures* of Robinson's framework. Meanwhile, the kind of set-theoretic or type-theoretic ontology necessary to make Robinson's framework acceptable to a modern rigorous mind is certainly nowhere to be found in the classical infinitesimalists, any more than Cantorian sets or Weierstrassian epsilon-ontics.

What we argue is *not* that Robinson is the necessary logical conclusion of Leibniz. Rather, we have argued, with Felix Klein, that there is a parallel B-track for the development of analysis that is often underestimated in traditional A-track epsilon-ontist historiography. The point is *not* that the B-track leads to Robinson, but rather than B-track is distinct from A-track; in fact, a number of modern infinitesimal theories could potentially fit the bill (for more details on the two tracks see Section 8).

The ultrapower construction is related to the classical construction of the reals starting from the set  $\mathbb{Q}^{\mathbb{N}}$  of sequences of rationals, in the following sense. In the classical construction, one works with the subset of  $\mathbb{Q}^{\mathbb{N}}$  consisting of Cauchy sequences. Meanwhile, in the ultrapower construction one works with *all* sequences (see Section 8). The researchers working in Robinson’s framework have often emphasized the need to learn model theory truly to understand infinitesimals. There may be some truth in this, but the point may have been overstated. A null sequence (i.e., sequence tending to zero) generates an infinitesimal, just as it did in Cauchy’s *Cours d’Analyse* (Cauchy 1821 [21]). The link between Cauchy’s infinitesimals and those in the ultrapower construction was explored by Sad, Teixeira & Baldino [76] and Borovik & Katz [16].

In the sequel whenever we use the adjective *intuitionistic*, we refer to mathematical frameworks relying on intuitionistic logic (rather than any specific system of *Intuitionism* such as Brouwer’s). Our intuitionistic framework is not BISH but rather that of Troelstra and van Dalen [87]. In a typical system relying on intuitionistic logic, one finds counterexamples that seem “paradoxical” from the classical viewpoint, such as Brouwerian counterexamples. Such examples are present in Bishop’s framework as well, though the traditional presentations thereof by Bridges and others tend to downplay this (see a more detailed discussion in Section 5). Feferman noted that

[Bishop] finesses the whole issue of how one arrives at Brouwer’s theorem [to the effect that every function on a closed interval is uniformly continuous] by saying that those are the only functions, at least initially, that one is going to talk about (Feferman 2000 [28]).

One of the themes of this text is the idea that, inspite of the incompatibility between the two frameworks for EVT, important insight about the problem can be gained from both. Our approach is both complementary and orthogonal to that of Wattenberg (1988 [88]).

## 2. GRADES OF CLARITY ACCORDING TO PEIRCE

Is there anything unclear about the extreme value theorem (EVT)? Or rather, as an intuitionist might put it, is there anything *clear* about the classical formulation of the EVT?

The clarity referred to in our title alludes to the Peircean analysis of the 3 grades (stages) of clarity in the emergence of a new concept. The framework was proposed by C. S. Peirce in 1897, in the context of his analysis of the concept of continuity and continuum, which, as he felt

at the time, is composed of infinitesimal parts (see [38, p. 103]). Peirce identified three stages in creating a novel concept:

there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience . . . The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable . . . The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems (Peirce 1897 [69] cited by Havenel 2008 [38, p. 87]).

The “three grades” can therefore be summarized as follows:

- (1) familiarity through experience;
- (2) abstract definition with an eye to future applications;
- (3) fruitful reasoning “made to turn” upon it, with applications.

The classical EVT asserts that a continuous function on a compact interval has a maximum. Attaining clarity in the matter of the EVT is therefore contingent upon

- clarity in the matter of *continuity*; and
- clarity in the matter of *maximum*.

We will analyze these issues in Section 3.

### 3. PERCEPTUAL CONTINUITY

Freudenthal [29] wrote that Cauchy invented our notion of continuity. Yet 60 years earlier, Abraham Gotthelf Kaestner (1719–1800) had the following to say in this matter:<sup>2</sup>

In a sequence<sup>3</sup> of quantities the increase or the decrease takes place in accordance with the law of continuity (*lege continui*), if after each term of the sequence, the one that follows or preceeds the given term, differs from it by as little as one wishes, in such a way that the difference of two consecutive terms may be less than any given quantity (Kaestner 1760 [42, paragraph 322, p. 180]).

Similar formulations can be found even earlier in Leibniz. Kaestner’s formulation of continuity is free of both infinitesimal language and  $\epsilon, \delta$

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<sup>2</sup>Translation kindly provided by D. Spalt.

<sup>3</sup>We have replaced the term *series* found in Kaestner, by the term *sequence*, to conform with modern usage.

terminology, yet it is a concise expression of a perceptual idea of continuity, at stage 1 of the Peircean ladder. Kaestner’s definition is far from being a mathematically coherent one. Bolzano and Cauchy retain priority even after this Kaestner is discovered. But what is interesting about his definition is its *local* nature. Contrary to popular belief, Bolzano and Cauchy were not the first to envision a local definition of continuity.

Decades later, Cauchy expressed perceptual continuity even more succinctly, in terms of a function varying by *imperceptible degrees* under minute changes of the independent variable (see, e.g., his letter to Coriolis of 1837 cited by Bottazzini [17, p. 107]).

At the perceptual level, the EVT seeks to capture the intuitively appealing idea of aiming for the “highest peak on the graph”.

Bolzano and Cauchy were the first ones to sense a need to provide proof of assertions such as the intermediate value theorem (IVT). Bolzano similarly proved the EVT in 1817 but his manuscript was only rediscovered in the 1860s. What kind of proof Cauchy (1789–1857) might have provided for the EVT we will never know. The EVT is usually attributed to Weierstrass who proved it in a course during 1861 (see [33, p. 18] and [33, p. 25, note 8]).

The presence of potentially infinitely many points in the domain of the function signals an immediate difficulty, residing in the possibility of the graph rising higher and higher without bound.

One begins to perceive the complexity of the task with the realisation of an inherent instability of an elusive “highest point” on the graph. Namely, the values at a pair of far-away points may be so close together as to make the choice difficult. A minute perturbation of the function may have an appreciable effect upon the answer. We summarize the two difficulties signaled so far:

- problem of existence of supremum over infinite domain;
- inherent instability of solution due to discontinuous dependence on data.

A perceptual approach to looking for a solution would be to subdivide the domain by means of a partition

$$\{x_i\} \tag{3.1}$$

so fine as to challenge the resolution of even the most powerful physical microscope. The number of partition points being finite, there is necessarily one,

$$x_{i_0},$$

with the highest value of the function among the partition points. Continuing the perceptual analysis, one now “zooms in” on the atom (or quark, or string)

$c$

carrying the partition point  $x_{i_0}$ . Since, by hypothesis, the function varies by imperceptible degrees, it does not deviate away from the value  $f(x_0)$  appreciably, on  $c$ .

Of course, if the point is in a visible picture as seen by the human eye, what looks like the maximum point may be one of several whose height is visibly the same, but one does not know which of these, at a higher magnification, will be the actual maximum. Thus, one may be zooming in on the wrong place. Nonetheless, in the perceptual stage being discussed here, one is *only* looking for what is visibly a maximum. Perceptually speaking, one will never know it’s the wrong place.

It is this phenomenon that makes it difficult to give a perceptual proof that can be turned into a mathematical proof of the EVT. If one draws the curve, one may be able to see the maximum, but a mechanical process of computing where the maximum is, that works without the human eye, is more subtle, perhaps even impossible.

How is such a perceptual proof transformed into a mathematical argument of a higher Peircean grade of clarity?

#### 4. CONSTRUCTIVE CLARITY

A constructive presentation of the EVT may be found in the text by Troelstra and van Dalen [87, p. 294-295]. A majority of the wider mathematical public is not intimately familiar with this approach. Therefore we will recall the main points in some detail. When a step in a constructive argument is identical with the classical one, we will preface it by an editorial clause “as usual”.

The starting point is a challenge to the non-constructive nature of “existence” proofs in classical mathematics. Such proofs generally go under the name of *proof by contradiction*. The main ingredient in a proof by contradiction is the law of excluded middle (LEM).

To provide an elementary example, consider the proof of irrationality of the square root of 2, as discussed by E. Bishop [12, p. 18]. Constructively speaking, being *irrational* is a stronger property than simply being *not rational*. Namely, *irrationality* involves an explicit lower bound for the distance from any rational (in terms of its denominator). Thus, for each rational  $m/n$ , the integer  $2n^2$  is divisible by an odd power of 2, while  $m^2$  is divisible by an even power of 2. Hence we have  $|2n^2 - m^2| \geq 1$ . Here we have applied LEM (or more precisely

the law of trichotomy) to an effectively decidable predicate over  $\mathbb{Z}$ . Since the decimal expansion of  $\sqrt{2}$  starts with  $1.41\dots$ , we may assume  $\frac{m}{n} \leq 1.5$ . It follows that

$$|\sqrt{2} - \frac{m}{n}| = \frac{|2n^2 - m^2|}{n^2 (\sqrt{2} + \frac{m}{n})} \geq \frac{1}{n^2 (\sqrt{2} + \frac{m}{n})} \geq \frac{1}{3n^2},$$

yielding a numerically meaningful proof of irrationality which avoids the use of LEM in its classical form (see Section 5 for a more advanced discussion). This is, of course, a special case of Liouville's theorem on diophantine approximation of algebraic numbers; see [36].

The intuitionist/constructivist challenge to classical mathematics has traditionally targeted the logical principle expressed by LEM. An additional point to keep in mind is a parallel change in the interpretation of the existential quantifier  $\exists$ , which now takes on a constructive meaning. Thus, to show that

$$\exists x : P(x),$$

one needs to specify an effective procedure for exhibiting such an  $x$ .

We say that  $x$  is the *supremum* of a set  $X \subset \mathbb{R}$  if and only if the following condition is satisfied:

$$(\forall y \in X) (y \leq x) \wedge (\forall k \in \mathbb{N}) (\exists y \in X) (y > x - 2^{-k}).$$

As usual, the supremum of a function  $f$  on  $[a, b]$  is the sup of the set  $\{f(x) : x \in [a, b]\}$ . A set  $X \subset \mathbb{R}$  is *totally bounded* if for each  $k \in \mathbb{N}$ , there is a finite collection of points  $x_0, \dots, x_{n-1} \in X$  such that

$$(\forall x \in X) (\exists i < n) (|x - x_i| < 2^{-k}).$$

**Lemma 4.1.** *A totally bounded set  $X \subset \mathbb{R}$  has a supremum.*

*Proof.* The proof exploits a constructive version of the notion of a Cauchy sequence. Namely, one requires  $|x_k - x_{k+n}| < 2^{-k}$  for every  $n \in \mathbb{N}$  (here  $n$  is independent of  $k$ ). For each  $k$  we specify a corresponding finite collection of points  $x_{k,0}, \dots, x_{k,n-1} \in X$  (with  $n = n(k)$ ) such that  $(\forall x \in X) (\exists i < n(k)) (|x - x_{k,i}| < 2^{-k})$ . Next, let

$$x_k := \max\{x_{k,i} : i < n(k)\}.$$

It follows that

$$(\forall x \in X) (\forall k) (x - x_k < 2^{-k}), \tag{4.1}$$

and we apply formula (4.1) to complete the proof.<sup>4</sup>  $\square$

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<sup>4</sup>In more detail, formula (4.1) applied to  $x = x_{k+1}$  implies  $x_{k+1} - x_k < 2^{-k}$ . Applying the same formula with  $k+1$  in place of  $k$  to  $x = x_k$  implies  $x_k - x_{k+1} < 2^{-k-1}$ . Hence  $|x_k - x_{k+1}| < 2^{-k}$ . Summing a geometric series we obtain  $|x_k - x_{k+n}| < 2^{-k} + 2^{-k-1} + \dots + 2^{-k-n-1} < 2^{-k+1}$ . The sequence  $\langle x_n : n \in \mathbb{N} \rangle$  is therefore Cauchy. One easily shows that its limit equals  $\sup X$ .



Compared with the classical version of the EVT, the constructive version requires a stronger hypothesis of *uniform* continuity,<sup>5</sup> and yields a weaker result, namely the existence of  $\sup(f)$  but not the existence of a maximum. Of course, the advantage of the constructive version is a clarification of the nature of constructively provable results, and may be considered more “honest”.

**Theorem 4.2** (Constructive EVT). *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be uniformly continuous. Then  $\sup(f)$  exists.*

*Proof.* Let  $\alpha$  be a modulus for  $f$ , so that one has

$$(\forall x, y \in [0, 1]) (\forall k) (|x - y| < 2^{-\alpha k} \implies |f(x) - f(y)| < 2^{-k}).$$

Then the set  $\{f(x) : x \in [0, 1]\}$  is totally bounded. Indeed, let  $n > 2^{\alpha k}$ . Let  $x_i := \frac{i}{n}$  for  $i < n$ . Then by uniform continuity, we obtain that  $(\forall x \in [0, 1]) (\exists i < n) (|f(x) - f(x_i)| < 2^{-k})$ , and therefore the set of values is totally bounded. The proof is completed by applying Lemma 4.1.  $\square$

The constructive impossibility of strengthening the conclusion of the theorem is discussed in Section 5.

## 5. COUNTEREXAMPLE TO THE EXISTENCE OF A MAXIMUM

The conclusion of Theorem 4.2 cannot be strengthened to the existence of a maximum, in the sense of the constructive formula

$$(\exists x \in [0, 1]) (f(x) = \sup(f)).$$

Indeed, let  $a$  be any real such that  $a \leq 0$  or  $a \geq 0$  is unknown. Next, define  $f$  on  $[0, 1]$  by setting  $f(x) = ax$ . Then  $\sup(f)$  is simply  $\max(0, a)$ , but the point where it is attained cannot be captured constructively (see [87, p. 295]). To elaborate on the foundational status of this example, note that the law of excluded middle:

$$P \vee \neg P$$

(“either  $P$  or (not  $P$ )”), is the strongest principle rejected by constructivists. A weaker principle is the LPO (limited principle of omniscience). The LPO is the main target of Bishop’s criticism in [13]. The LPO is formulated in terms of sequences, as the principle that it is possible to search “a sequence of integers to see whether they all vanish” [13, p. 511]. The LPO is equivalent to the law of trichotomy:

$$(a < 0) \vee (a = 0) \vee (a > 0).$$

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<sup>5</sup>Pointwise continuity and uniform continuity on a compact interval are equivalent with respect to classical logic, but not with respect to intuitionistic logic.

An even weaker principle is  $(a \leq 0) \vee (a \geq 0)$ . The existence of  $a$  satisfying the negation

$$\neg((a \leq 0) \vee (a \geq 0))$$

is exploited in the construction of the counterexample under discussion. The principle  $(a \leq 0) \vee (a \geq 0)$  is false intuitionistically. After discussing real numbers  $x \geq 0$  such that it is “*not*” true that  $x > 0$  or  $x = 0$ , Bishop writes:

In much the same way we can construct a real number  $x$  such that it is *not* true that  $x \geq 0$  or  $x \leq 0$  (Bishop 1967 [11, p. 26]), (Bishop & Bridges 1985 [14, p. 28]).

The fact that an  $a$  satisfying  $\neg((a \leq 0) \vee (a \geq 0))$  yields a counterexample  $f(x) = ax$  to the extreme value theorem (EVT) on  $[0, 1]$  is alluded to by Bishop in [11, p. 59, exercise 9]; [14, p. 62, exercise 11]. Bridges interprets Bishop’s italicized “*not*” as referring to a Brouwerian counterexample, and asserts that trichotomy as well as the principle  $(a \leq 0) \vee (a \geq 0)$  are independent of Bishopian constructivism. See Bridges [19] for details; a useful summary may be found in Taylor [86].

## 6. REUNITING THE ANTIPODES

The advantage of the constructive framework is the anchoring of the distinction between what can be exhibited and what cannot, in the very mathematical formalism. Rather than being an afterthought that may or may not trickle down to the students, the distinction is built into the intuitionistic hardware. Meanwhile, the level of detail required to operate such machinery risks masking the simple perceptual insights at the level of the cognitive underpinnings of the EVT.

Viewed as a *companion* to the classical approach, the intuitionistic framework can usefully enhance constructive issues that are otherwise relegated to footnotes, appendices, or optional material in textbooks. Meanwhile, viewed as an *alternative* to the classical approach, the intuitionistic framework risks masking important conceptual phenomena available in classical idealisations, particularly in areas such as geometry and mathematical physics. Minimal surfaces, geodesics, variational principles, etc., are inextricably tied in with ever more sophisticated implementations of the classical EVT, and rely upon the existence of the actual extremal points rather than merely suprema. Thus, the existence of the Calabi-Yau manifolds, ubiquitous in both differential geometry and mathematical physics, is nonconstructive; see Yau & Nadis [89].

More specifically, general relativity routinely exploits versions of the extreme value theorem, in the form of the existence of solutions to variational principles, such as geodesics, be it spacelike, timelike, or lightlike. At a deeper level, S.P. Novikov [67, 68] wrote about Hilbert’s meaningful contribution to relativity theory, in the form of discovering a Lagrangian for Einstein’s equation for spacetime. Hilbert’s deep insight was to show that general relativity, too, can be written in Lagrangian form, which is a satisfying conceptual insight.

A radical constructivist’s reaction would be to dismiss the material discussed in the previous paragraphs as relying on LEM (needed for the EVT), hence lacking numerical meaning, and therefore meaningless. In short, radical constructivism (as opposed to the liberal variety) adopts a theory of meaning amounting to an ostrich effect as far as certain significant scientific insights are concerned. A quarter century ago, M. Beeson already acknowledged constructivism’s problem with the calculus of variations in the following terms:

Calculus of variations is a vast and important field which lies right on the frontier between constructive and non-constructive mathematics (Beeson 1985 [5, p. 22]).

An even more striking example is the Hawking-Penrose singularity theorem explored by Hellman [39], which relies on fixed point theorems and therefore is also constructively unacceptable, at least in its present form. However, the singularity theorem does provide important scientific insight. Roughly speaking, one of the versions of the theorem asserts that certain natural conditions on curvature (that are arguably satisfied experimentally in the visible universe) force the existence of a singularity when the solution is continued backward in time, resulting in a kind of a theoretical justification of the Big Bang. Such an insight cannot be described as “meaningless” by any reasonable standard of meaning preceding nominalist commitments; see (Katz & Katz 2012 [43]) for more details.

## 7. KRONECKER AND CONSTRUCTIVISM

Kronecker subdivided mathematics into three fields: analysis, algebra, and number theory (or arithmetic). In his 1861 inaugural speech at the Academy of Science of Berlin, Kronecker said:

The study of complex multiplication of elliptic functions leading to works the object of which can be characterized as being drawn from analysis, motivated by algebra and driven by number theory (see Gauthier [30, p. 39]).

In his 1886 letter to Lipschitz, he declared that with the publication of his 1882 work *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, he has found the long-sought foundations of his theory of forms with the arithmetisation of algebra which had been the goal of his mathematical life.

In his 1891 lectures [59], Kronecker criticized Bolzano for having used the crudest means (*mit den rohesten Mitteln*) in his proof of the intermediate value theorem; see Gauthier (2009 [31, p. 225]), (2013 [32, p. 39]); Boniface & Schappacher [15, p. 269-270]).

Thus, Kronecker did not consider geometry and mathematical physics as part of mathematics. Of the three fields of analysis, geometry, and mathematical physics (see Boniface & Schappacher [15, p. 211]), he argued in favor of a constructive arithmetisation of analysis alone.<sup>6</sup> This was to be a reformulation of analysis in terms of the natural numbers rather than the speculative constructs such as  $\mathbb{R}$ , as envisioned by his contemporaries Cantor, Dedekind, and Weierstrass. Kronecker plainly acknowledged that two-thirds of what is today considered mathematics (namely, what he referred to as geometry and physics) is not amenable to such arithmetisation.<sup>7</sup> Radical constructivists attempting to enlist Kronecker to their cause of constructivizing *all* of modern mathematics may therefore be “more vigorous than accurate”, to quote Robinson (1968 [72]).

As a transition to the remainder of this text, we note that classical mathematics can be thought of as an extension of constructive mathematics, inasmuch as the former uses more and stronger axioms than the latter, although the verificationist interpretation of the existence quantifier necessarily leads to a clash with the classical viewpoint (see end of Section 5 for details). Meanwhile, analysis over the hyperreals can be done in the framework of the standard Zermelo-Fraenkel axiom system with the axiom of choice. The key consequence of the latter is the existence of ultrafilters proved by Tarski [85] in 1930 (the use of ultrafilters is explained in Subsection 8.4).

An admirable attempt to bridge the gap between “the constructive and nonstandard views of the continuum” resulted in a volume of publications [78], but not immediately in a unity of purpose. A type of proof described as “constructive modulo an ultrafilter” proposed by

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<sup>6</sup>The meaning of the term “arithmetisation” has changed since Kronecker introduced it, and today refers to the traditional set-theoretic foundations of  $\mathbb{R}$  in particular and analysis in general.

<sup>7</sup>H. Edwards, a contemporary expert on Kronecker’s thinking, has done much to restore balance in the popular perception of Kronecker (see e.g., [25]), but may have overlooked the three-fold partition and its ramifications for constructivisation.

Ross [73], [74] represents an interesting attempt at a dialogue. Our theme here is that, even though the two conceptual frameworks may not be compatible, both can yield useful mathematical insight.

An intriguing proposal for bridging the gap specifically in the context of the extreme value theorem was recently made by Schuster [80]. Here one exploits the principle of unique choice, alternatively called the principle of non-choice. The heuristic idea is that, if uniqueness is sufficiently ubiquitous (e.g., “if a continuous function on a complete metric space has approximate roots and, in a uniform manner, at most one root”), then existence follows, as well (“it actually has a root”) (see also [79]). There is a vast literature on the related uniqueness paradigm. Thus, in Kohlenbach [55], the uniform notion of uniqueness was introduced under the name “modulus of (uniform) uniqueness” (see also his book [56]). Some of these ideas were anticipated in a 1979 text by Kreinovich [57] (see also his review [58]). The theorem that if a computable function in  $C[0, 1]$  attains its maximum at a unique point then that point is computable, was already proved by Grzegorzczuk [34] in 1955. Further work includes Ishihara [41]; Berger and Ishihara [9]; Berger, Bridges, and Schuster [10]; Diener and Loeb [24]; Schwichtenberg [81]; Bridges [20]; and others.

## 8. INFINITESIMAL CLARITY

Approximation issues stressed in the constructive approach are important for both mathematics and its applications. Similarly, it is important for the student to realize that the tangent line is the limit of secant lines, which gives geometric motivation for the idea of derivative.

**8.1. Nominalistic reconstructions.** At the same time, it is important to realize that the Weierstrassian  $(\epsilon, \delta)$  approach tends to remove motion and geometry from the definition of basic concepts of the calculus. The game of “you tell me the epsilon, I will tell you the delta” superficially resembles approximation theory. However, the crucial issue is the functional dependence of  $\delta$  on  $\epsilon$ , rather than any specific approximation; no wonder engineering students, who are certainly vitally concerned with approximation, are seldom taught the  $(\epsilon, \delta)$  method. In reality dressing students to perform multiple-quantifier epsilontic logical stunts on pretense of teaching them infinitesimal calculus is merely a way of dressing up a bug to look like a feature (to borrow a quip from computer science folklore), as is apparent if one compares this approach to the lucidity of its infinitesimal counterpart.

The Weierstrassian  $(\epsilon, \delta)$  approach was necessitated by an inability to justify the ontological material that naturally arose in scientific inquiry

during the 17th, 18th, and 19th centuries, namely the infinitesimals, which were present at the conception of the theory by Leibniz, Johann Bernoulli, and others. For more details, see Bair et al. [1]; Bottazzi et al. [4]; Mormann et al. [66].

Starting in the 1870s, Cantor, Dedekind, and Weierstrass justified the logical complications they introduced into the foundations, in terms of their success in eliminating foundational material they were unable to justify. Cantor went as far as calling infinitesimals the “cholera bacillus of mathematics”, and published a paper purporting to “prove” that they are self-contradictory; see Ehrlich (2006 [27]) for more details. Their work amounted to a nominalistic reconstruction of analysis, by eliminating ontological material they could not account for; see Katz & Katz (2012 [43]) for more details. The success of their reconstruction resulted in a widespread ad-hoc ontological commitment to an exclusive reality of the real numbers (or their constructive analogues).

Such a reductive philosophical commitment has taken a toll on the development of mathematics. Thus, Cauchy’s Dirac delta function and its applications in Fourier analysis were forgotten for over a century, because of the reductive ideology that eliminated infinitesimals, without which Cauchy’s applications of what would later be called the Dirac delta function could not be sustained; see Laugwitz (1989 [60]); Katz & Tall (2013 [47]). Similarly, Cauchy was the one who invented our notion of continuity, and he defined it in terms of infinitesimals. To Cauchy, an infinitesimal was generated by a null sequence. This is related both perceptually and formally to the construction of infinitesimals in the ultrapower approach (see [76]).

An infinitesimal-enriched continuum offers a possibility of mimicking more closely the perceptual analysis of the EVT, in constructing a formal proof, due to the availability of a hierarchical number system, with an Archimedean continuum (A-continuum for short) englobed inside an infinitesimal-enriched Bernoullian continuum (B-continuum for short).

**8.2. Klein on rivalry of continua.** Felix Klein described a rivalry of such continua in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that

The scientific mathematics of today is built upon the series of developments which we have been outlining. But *an essentially different conception of infinitesimal calculus has been running parallel with this [conception]*

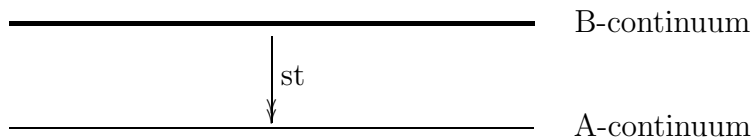


FIGURE 1. Taking standard part

*through the centuries* (Klein [51, p. 214]) [emphasis added—authors].

Such a different conception, according to Klein,

harks back to old metaphysical speculations concerning the *structure of the continuum* according to which this was made up of [...] infinitely small parts [51, p. 214] [emphasis added—authors].

Victor J. Katz (2014 [48]) appears to imply that a B-track approach based on notions of infinitesimals or indivisibles is limited to “the work of Fermat, Newton, Leibniz and many others in the 17th and 18th centuries”. This does not appear to be Klein’s view. Klein formulated a condition, in terms of the mean value theorem, for what would qualify as a successful theory of infinitesimals, and concluded:

I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive (Klein 1908 [51, p. 219]).

Klein was referring to the current work on infinitesimal-enriched systems by Levi-Civita, Bettazzi, Stolz, and others. In Klein’s mind, the infinitesimal track was very much a current research topic; see Ehlich (2006 [27]) for a detailed coverage of the work on infinitesimals around 1900.

**8.3. Formalizing Leibniz.** Leibniz’s approach to the differential quotient<sup>8</sup>

$$\frac{dy}{dx}$$

(today called the derivative) was formalized by Robinson. Here one exploits a map called *the standard part*, denoted “st”, from the finite part of a B-continuum, to the A-continuum, as illustrated in Figure 1.

In the context of the hyperreal extension of the real numbers, the map st “rounds off” each finite hyperreal  $x$  to the nearest real  $x_0 = \text{st}(x) \in \mathbb{R}$ . In other words, the map “st” collapses the cluster of points

<sup>8</sup>See (Sherry 1987 [82]) and (Katz & Sherry 2012 [46], [45]).

infinitely close to a real number  $x_0$ , back to  $x_0$ . Note that both the term “hyper-real field”, and an ultrafilter construction thereof, are due to E. Hewitt in 1948 (see [40, p. 74]). The transfer principle allowing one to extend every first-order real statement to the hyperreals, is due to J. Łoś in 1955 (see [65]). Thus, the Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle. See (Keisler 1994 [50]) for a historical outline.

We will denote such a B-continuum by the new symbol  $\mathbb{I}\mathbb{R}$ . We will also denote its finite (limited) part by

$$\mathbb{I}\mathbb{R}_{<\infty}.$$

The map “st” sends each finite point  $x \in \mathbb{I}\mathbb{R}$ , to the real point  $\text{st}(x) \in \mathbb{R}$  infinitely close to  $x$ :

$$\begin{array}{c} \mathbb{I}\mathbb{R}_{<\infty} \\ \downarrow \text{st} \\ \mathbb{R} \end{array}$$

We illustrate the construction by means of an infinite-resolution microscope in Figure 2.

Robinson defined the derivative as  $\text{st}\left(\frac{\Delta y}{\Delta x}\right)$ , instead of  $\Delta y/\Delta x$ . For an accessible exposition (see H. J. Keisler [49, 50]).

**8.4. Ultrapower.** To elaborate on the ultrapower construction of the hyperreals, let  $\mathbb{Q}^{\mathbb{N}}$  denote the space of sequences of rational numbers. Let  $(\mathbb{Q}^{\mathbb{N}})_C$  denote the subspace consisting of Cauchy sequences. The reals are by definition the quotient field  $\mathbb{R} := (\mathbb{Q}^{\mathbb{N}})_C / \mathcal{F}_{\text{null}}$ , where  $\mathcal{F}_{\text{null}}$  contains all null sequences. Meanwhile, the hyperreals can be obtained by forming the quotient  $\mathbb{I}\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \mathcal{F}_u$ , where a sequence  $\langle u_n \rangle$  is in  $\mathcal{F}_u$  if and only if the set  $\{n \in \mathbb{N} : u_n = 0\}$  is a member of a fixed ultrafilter. To give an example, the sequence  $\left\langle \frac{(-1)^n}{n} : n \in \mathbb{N} \right\rangle$  represents a nonzero infinitesimal, whose sign depends on whether or not the set  $2\mathbb{N}$  is a member of the ultrafilter. A helpful “semicolon” notation for presenting an extended decimal expansion of a hyperreal was described by Lightstone [63].

## 9. HYPERREAL EXTREME VALUE THEOREM

We now return to the EVT. The classical proof of the EVT usually proceeds in two or more stages. Typically, one first shows that the function is bounded. Then one proceeds to construct an extremum by one or another procedure involving choices of sequences. The hyperreal



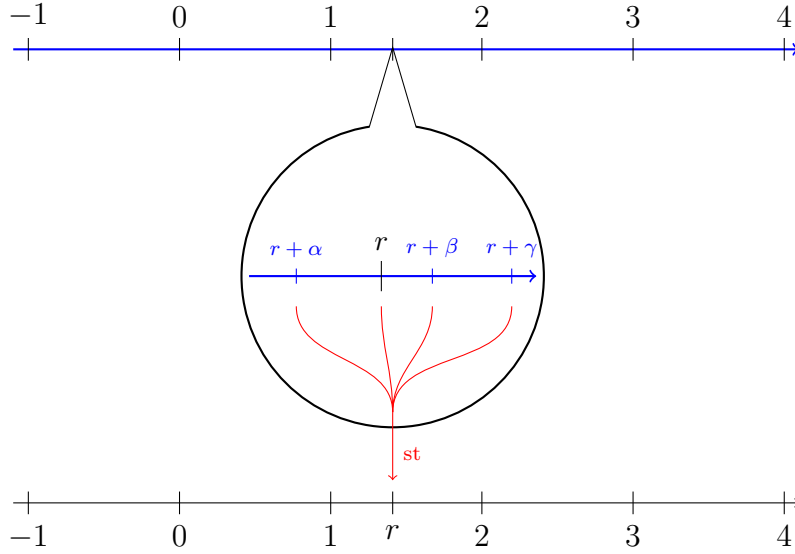


FIGURE 2. The standard part function,  $st$ , “rounds off” a finite hyperreal to the nearest real number. The function  $st$  is here represented by a vertical projection. An “infinitesimal microscope” is used to zoom in on an infinitesimal neighborhood of a standard real number  $r$ , where  $\alpha$ ,  $\beta$ , and  $\gamma$  represent typical infinitesimals. Courtesy of Wikipedia.

approach is both more economical (there is no need to prove boundedness first) and less technical. To show that a continuous function  $f(x)$  on  $[0, 1]$  has a maximum, let

$$H$$

be an infinite hypernatural number (for instance, the one represented by the sequence  $\langle 1, 2, 3, \dots \rangle$  with respect to the ultrapower construction outlined in Subsection 8.4). The real interval  $[0, 1]$  has a natural hyperreal extension. Consider its partition into  $H$  subintervals of equal infinitesimal length  $\frac{1}{H}$ , with partition points  $x_i = \frac{i}{H}$  as  $i$  “runs” from 0 to  $H$ . The existence of such a partition follows by the transfer principle (see more below) applied to the first order formula

$$(\forall n \in \mathbb{N}) (\forall x \in [0, 1]) (\exists i < n) \left( \frac{i}{n} \leq x < \frac{i+1}{n} \right).$$

The function  $f$  is naturally extended to the hyperreals between 0 and 1. Note that in the real setting (when the number of partition points is finite), a point with the maximal value of  $f$  can always be chosen among the partition points  $x_i$ , by induction. We have the following first order property expressing the existence of a maximum of  $f$  over a

finite collection:

$$(\forall n \in \mathbb{N}) (\exists i_0 < n) (\forall i < n) \left( f\left(\frac{i_0}{n}\right) \geq f\left(\frac{i}{n}\right) \right).$$

We now apply the transfer principle to obtain

$$(\forall H \in \mathbb{HN}) (\exists i_0 < H) (\forall i < H) \left( f\left(\frac{i_0}{H}\right) \geq f\left(\frac{i}{H}\right) \right), \quad (9.1)$$

where  $\mathbb{HN}$  is the collection of hypernatural numbers. Formula (9.1) is true in particular for a particular infinite hypernatural  $H \in \mathbb{HN} \setminus \mathbb{N}$ . Thus, there is a hypernatural  $i_0$  such that  $0 \leq i_0 \leq H$  and

$$f(x_{i_0}) \geq f(x_i) \quad (9.2)$$

for all  $i = 0, \dots, H$ . Consider the real point  $c = \text{st}(x_{i_0})$  where “st” is the standard part function. By continuity of  $f$ , we have  $f(x_{i_0}) \approx f(c)$ , and therefore

$$\text{st}(f(x_{i_0})) = f(\text{st}(x_{i_0})) = f(c).$$

An arbitrary real point  $x$  lies in a suitable sub-interval of the partition, namely  $x \in [x_i, x_{i+1}]$ , so that  $\text{st}(x_i) = x$ . Applying “st” to inequality (9.2), we obtain

$$\text{st}(f(x_{i_0})) \geq \text{st}(f(x_i)).$$

Hence  $f(c) \geq f(x)$ , for all real  $x$ , proving  $c$  to be a maximum of  $f$ . Note that the argument follows closely our *perceptual* analysis in Section 3.

## 10. APPROACHES AND INVITATIONS

Robinson’s approach in (Robinson 1966 [71]) was formulated in the framework of model theory of mathematical logic.

Two decades later, Lindström (1988 [64]) presented an alternative analytical approach in his text *An invitation to nonstandard analysis*, described as follows:

I have tried to make the subject look the way it would had it been developed by analysts or topologists and not logicians. This is the explanation for certain unusual features such as my insistence on working with ultra-power models and my willingness to downplay the importance of first order languages (Lindstrom [64, p. 1]).

In this approach, the proof of EVT, though essentially the same, is even more elementary and short, because one does not need to use the transfer principle.

## 11. CONCLUSION

The strength of the constructive approach is the ability to place into sharp relief a hard-nosed analysis of what can be effectively exhibited, and what cannot. The strength of the infinitesimal approach is its closer fit with the perceptual analysis of the phenomenon at the heart of the EVT. The complementarity of the resulting insights ultimately points to a companionship, rather than a rivalry, between the two approaches. Such fruitful complementarity persists inspite of possible formal incompatibilities of the intuitionistic and the classical frameworks.

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